

# SYMMETRIC SCHRÖDER PATHS AND RESTRICTED INVOLUTIONS

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**ABSTRACT.** Let  $\mathcal{A}_k$  be the set of permutations in the symmetric group  $\mathfrak{S}_k$  with prefix 12. This paper concerns the enumeration of involutions which avoid the set of patterns  $\mathcal{A}_k$ . We present a bijection between symmetric Schröder paths of length  $2n$  and involutions of length  $n + 1$  avoiding  $\mathcal{A}_4$ . Statistics such as the number of right-to-left maxima and fixed points of the involution correspond to the number of steps in the symmetric Schröder path of a particular type. For each  $k \geq 3$  we determine the generating function for the number of involutions avoiding the subsequences in  $\mathcal{A}_k$ , according to length, first entry and number of fixed points.

## 1. INTRODUCTION

Let  $\mathfrak{S}_n$  denote the set of permutations of  $[n] = \{1, \dots, n\}$ , written in one-line notation. For two permutations  $\pi \in \mathfrak{S}_n$  and  $\tau \in \mathfrak{S}_k$ , an *occurrence* of  $\tau$  in  $\pi$  is a subsequence  $1 \leq i_1 < i_2 < \dots < i_k \leq n$  such that  $(\pi_{i_1}, \dots, \pi_{i_k})$  is order-isomorphic to  $\tau$ ; in such a context  $\tau$  is usually called a *pattern*. We say that  $\pi$  *avoids*  $\tau$ , or is  $\tau$ -*avoiding*, if there is no occurrence of  $\tau$  in  $\pi$ . A natural generalization of single pattern avoidance is *subset avoidance*, that is, we say that  $\pi \in \mathfrak{S}_n$  avoids a subset  $T \subseteq \mathfrak{S}_k$  if  $\pi$  avoids all  $\tau \in T$ . The set of all  $\tau$ -avoiding (resp.  $T$ -avoiding) permutations of length  $n$  is denoted  $\mathfrak{S}_n(\tau)$  (resp.  $\mathfrak{S}_n(T)$ ).

Several authors have considered the case of general  $k$  in which  $T$  enjoys various algebraic properties. Barucci et al. [2] treat the case of permutations avoiding the collection of permutations in  $\mathfrak{S}_k$  that have suffix  $(k - 1)k$ . Adin and Roichman [1] look at the case where  $T$  is a Kazhdan–Lusztig cell of  $\mathfrak{S}_k$ , or, equivalently, a Knuth equivalence class (see [12, Vol. 2, Ch. A1]). Mansour and Vainshtein [10] consider the situation where  $T$  is a maximal parabolic subgroup of  $\mathfrak{S}_k$ .

In the current paper an analogous result is established for pattern-avoiding involutions. We say  $\pi$  is an *involution* whenever  $\pi_{\pi_i} = i$  for all  $i \in [n]$ . Let  $\mathcal{I}_n$  denote the set of involutions of  $[n]$ . The set of all  $\tau$ -avoiding (resp.  $T$ -avoiding) involutions of length  $n$  is denoted  $\mathcal{I}_n(\tau)$  (resp.  $\mathcal{I}_n(T)$ ).

Simion and Schmidt [11] considered the first cases of pattern-avoiding involutions, which was continued in Gouyou-Beauchamps [5] and Gessel [6] for increasing patterns, and subsequently in Guibert's Ph.D. thesis [7]. This paper concerns the enumeration of involutions which avoid the class of permutations in  $\mathfrak{S}_k$  with prefix 12, that is,

$$\mathcal{A}_k = \{\pi_1 \pi_2 \dots \pi_k \in \mathfrak{S}_k \mid \pi_1 = 1, \pi_2 = 2\}.$$

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We denote by  $I_n$  the cardinality of the set  $\mathcal{I}_n$ . We say that  $i$  is a fixed point of a permutation  $\pi$  if  $\pi_i = i$ . Define  $J_n(p)$  to be the polynomial  $\sum_{j=0}^n I_{n;j} p^j$ , where  $I_{n;j}$  is the number of involutions in  $\mathcal{I}_n$  with  $j$  fixed points. For example  $J_3(p) = 3p + p^3$ . It is not hard to see that the polynomials  $J_n(p)$  satisfy the recurrence relation  $J_n(p) = pJ_{n-1}(p) + (n-1)J_{n-2}(p)$ ,  $n \geq 2$ , with the initial conditions  $J_0(p) = 1$  and  $J_1(p) = p$ . The exponential generating function for the sequence  $\{J_n(p)\}_{n \geq 0}$  is given by  $e^{px+x^2/2}$ .

The main result of this paper can be formulated as follows.

**Theorem 1.1.** *Let  $k \geq 2$ . The generating function for the number of  $\mathcal{A}_k$ -avoiding involutions of length  $n$  is given by*

$$\begin{aligned} \sum_{n \geq 0} \sum_{\pi \in \mathcal{I}_n(\mathcal{A}_k)} x^n p^{\#\text{fixed points}(\pi)} = \\ \sum_{j=0}^{k-3} J_j(p) x^j - \frac{x^{k-3}}{2} \left( p + (p(k-3)x^2 - 2x - p)u_0(x) \right) J_{k-2}(p) \\ - \frac{x^{k-4}}{2} \left( x + p - (x^3(k-3) - px^2(k-1) + x + p)u_0(x) \right) J_{k-3}(p), \end{aligned}$$

where  $u_0(x) = 1/\sqrt{1 - 2(k-1)x^2 + (k-3)^2x^4}$ .

The proof is given in Section 3. Theorem 1.1 with  $k = 3$  and  $p = 1$  shows the generating function for the number 123-avoiding involutions of length  $n$  to be  $\frac{2x}{2x-1+\sqrt{1-4x^2}} = \sum_{n \geq 0} \binom{n}{\lfloor n/2 \rfloor} x^n$  (see [11]). Also, Theorem 1.1 with  $k = 3$  and  $p = 0$  gives the number of 123-avoiding involutions of length  $2n$  without fixed points to be  $\frac{1}{2} \binom{n}{\lfloor n/2 \rfloor}$ . Moreover, Theorem 1.1 with  $k = 4$  and  $p = 1$  gives the generating function for the number  $\{1234, 1243\}$ -avoiding involutions of length  $n$  to be  $\frac{1-x}{2} + \frac{1+x}{2} \sqrt{\frac{1+2x-x^2}{1-2x-x^2}}$ . In Section 2 we present a bijection between symmetric Schröder paths of length  $n-1$  and  $\{1234, 1243\}$ -avoiding involutions of length  $n$ , thereby providing a combinatorial proof of the above result with  $k = 4$ .

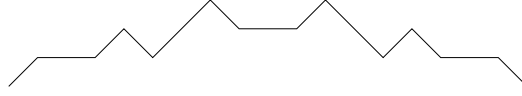
## 2. SYMMETRIC SCHRÖDER PATHS AND $\{1234, 1243\}$ -AVOIDING INVOLUTIONS

A *Schröder path* of length  $2n$  is a lattice path from  $(0,0)$  to  $(2n,0)$  consisting of double horizontal steps  $\mathbf{h} = (2,0)$ , up steps  $\mathbf{u} = (1,1)$  and down steps  $\mathbf{d} = (1,-1)$  that never goes below the  $x$ -axis. The set of all Schröder paths of length  $2n$  is enumerated by the  $n$ -th Schröder number. Kremer [8, Corollary 9] showed that  $\mathfrak{S}_n(1243, 2143)$  is also enumerated by the  $n$ -th Schröder number.

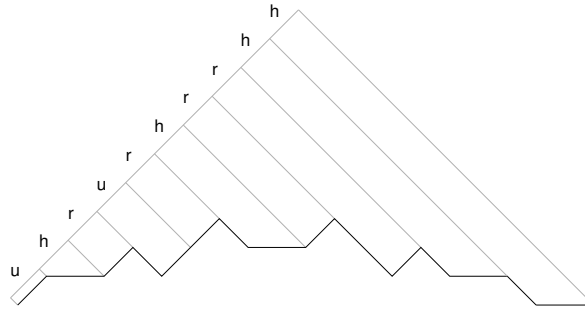
A Schröder path of length  $2n$  is called *symmetric* if it is symmetric about the line  $x = n$ . Let  $\text{Sh}_n$  be the collection of all such paths. In this section we give a bijection between *symmetric Schröder paths* of length  $2n$  and the class of involutions in  $\mathfrak{S}_{n+1}$  that avoid the patterns  $\{1234, 1243\}$ .

We will now describe a map  $\phi : \text{Sh}_n \rightarrow \mathcal{I}_{n+1}(1234, 1243)$ . Alongside this description will be an example of the map acting on the path  $p = \text{uhuduudhuddudhd} \in \text{Sh}_9$ . Several points may

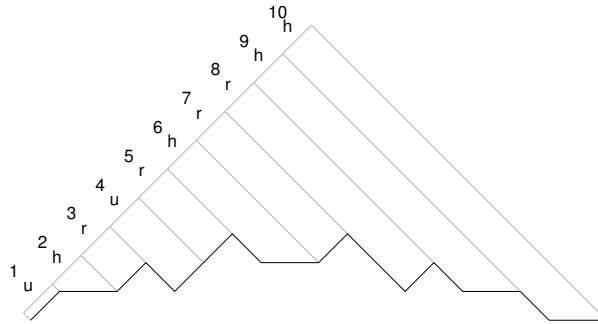
at first appear extraneous, however these will be required in the proof of the Theorem which follows.



Given  $p \in \text{Sh}_n$ , let  $p'$  be the word of length  $n + 1$  obtained from  $p$  by first appending a horizontal step  $h$  to the end, then replacing all occurrences of  $ud$  in  $p$  by  $r$ , and finally deleting all remaining  $d$ 's. This is equivalent to projecting the steps of the path onto the diagonal, and replacing any  $u$ 's that are followed by a  $d$  by  $r$ .



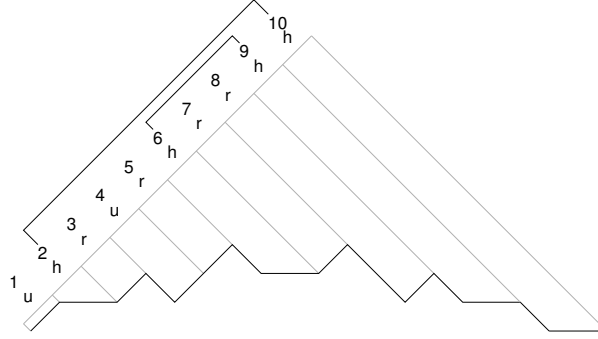
From the diagram the example path is  $p' = \text{uhrurhrrhh}$ . In general,  $p' = p_1 \dots p_{n+1} \in \{u, r, h\}^{n+1}$ . Now form the three sets  $A_h, A_r, A_u$  according to the rule  $i \in A_x$  if  $p_i = x$ .



In the example,  $A_h = \{2, 6, 9, 10\}$ ,  $A_r = \{3, 5, 7, 8\}$  and  $A_u = \{1, 4\}$ .

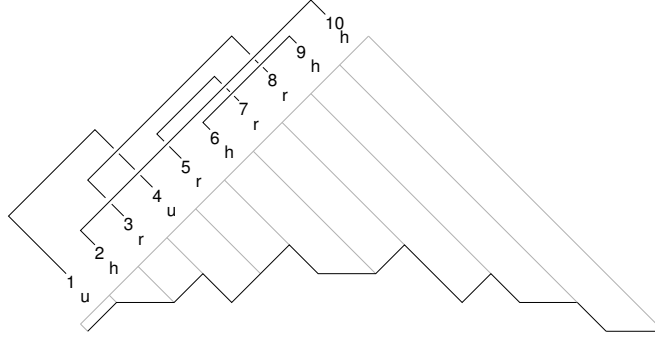
Starting with the largest entry in  $A_h$ , replace all  $h$ 's in  $p'$  (from left to right) with the entries from  $A_h$  so that this sequence is decreasing. This is equivalent to forming a sequence of transpositions (or a fixed point), the first of which will have as first entry the index of the first  $h$  in  $p'$  and whose second entry is the index of the last  $h$  in  $p'$ . The second transposition (or fixed point) will have as first entry the index of the second  $h$  in  $p'$  and whose second entry is the index of the second last  $h$  in  $p'$ , and so forth.

Since  $A_h = \{2, 6, 9, 10\}$  in the example, we have the transpositions  $(2, 10)$  and  $(6, 9)$ .



Do likewise for the sets  $A_r$  and  $A_u$ . The *label* of a particular h, r or u in  $p'$  is the value that replaces it. Call the resulting permutation  $\pi = \phi(p)$ , i.e. the labels of  $p'$  read from left to right.

The label of the first h in  $p'$  above is 10 and the label of the third h in  $p'$  above is 6. In the example, since  $A_h = \{2, 6, 9, 10\}$ ,  $A_r = \{3, 5, 7, 8\}$  and  $A_u = \{1, 4\}$ ,  $\pi = \phi(p)$  is the permutation (involution) with cycles  $(2, 10), (6, 9), (3, 8), (5, 7)$  and  $(1, 4)$ . Thus we have  $\phi(p) = (4, 10, 8, 1, 7, 9, 5, 3, 6, 2)$ .



Another way to see this construction is in terms of layers of right-to-left maxima. An element  $\pi_i$  of a permutation  $\pi$  is called a *right-to-left maximum* if it is greater than all elements that follow it, i.e.  $\pi_i > \pi_j$  for all  $j > i$ . We define successively the  $r$ -right-to-left maxima for a permutation  $\pi \in \mathfrak{S}_n$ . Let  $\pi^{(1)}$  be the word consisting of all elements of  $\pi$ . For  $r \geq 1$ , the right-to-left maxima of  $\pi^{(r)}$  are called  $r$ -right-to-left maxima of  $\pi$ . Let  $\pi^{(r+1)}$  be the subword obtained from  $\pi^{(r)}$  by removing all  $r$ -right-to-left maxima. For example, the permutation  $\pi = 674583912 \in \mathfrak{S}_9$  has the 1-right-to-left maxima 9 and 2; the 2-right-to-left maxima 8, 3 and 1; the 3-right-to-left maxima 7 and 5; and the 4-right-to-left maxima 6 and 4. Note that the  $r$ -right-to-left maxima of  $\pi$  form a decreasing subsequence for each  $r$ .

Let  $\pi$  be the unique permutation (in fact, it will be an involution) in  $\mathfrak{S}_{n+1}$  with 1 right-to-left maxima  $A_h$ , 2 right-to-left maxima  $A_r$  and 3 right-to-left maxima  $A_u$ . If  $A_h = \{x_1, \dots, x_\alpha\}$ ,  $A_r = \{y_1, \dots, y_\beta\}$  and  $A_u = \{z_1, \dots, z_\gamma\}$  then the cycles of  $\pi$  are

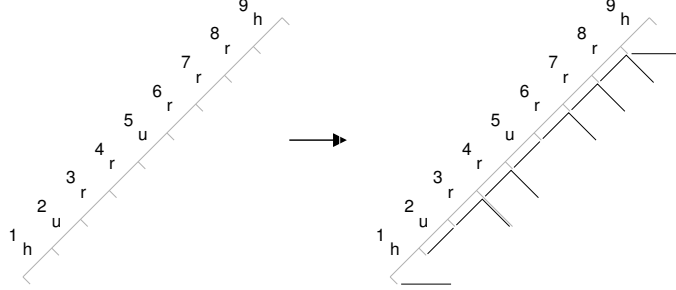
$$(x_1 x_\alpha) (x_2 x_{\alpha-1}) \cdots (y_1 y_\beta) (y_2 y_{\beta-1}) \cdots (z_1 z_\gamma) (z_2 z_{\gamma-1}) \cdots$$

We point out that if  $\gamma = 2m + 1$  then  $(x_{m+1})$  will be a fixed point. Consequently there will be at most three fixed points in the resulting involution.

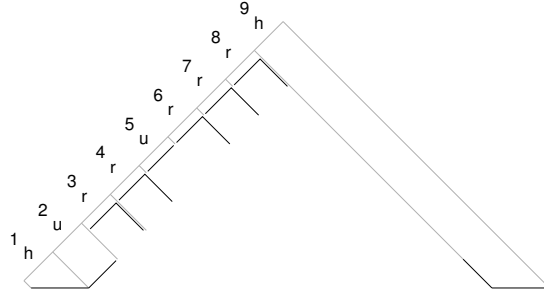
The inverse map  $\phi^{-1}$  is described by means of an example. Consider

$$\pi = (10, 5, 8, 7, 2, 6, 4, 3, 9, 1) \in \mathcal{I}_{10}(1234, 1243).$$

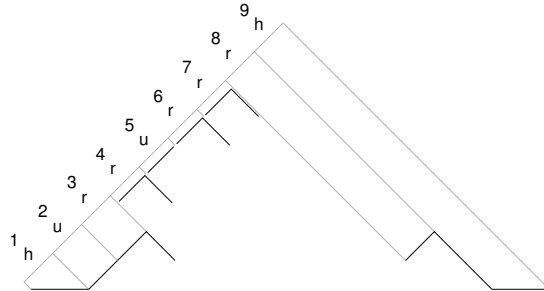
The sets of 1, 2 and 3 right-to-left maxima are  $A_h = \{1, 9, 10\}$ ,  $A_r = \{3, 4, 6, 7, 8\}$  and  $A_u = \{2, 5\}$ , respectively. This gives  $p' = \text{hurrurrrhh}$ . After removing the final h, we have



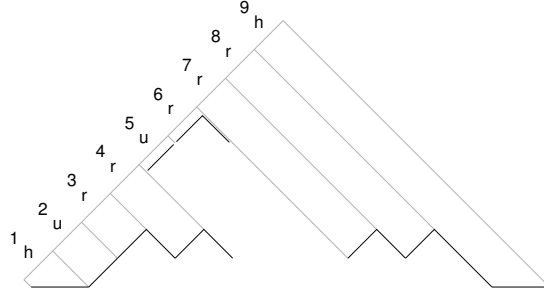
Beginning with the last step (at position 9), we push this down so that it is symmetric with the first entry. We then move the second step of  $p'$  down to meet the path. It is u so there must be a d inserted at the opposite end so the path is symmetric.



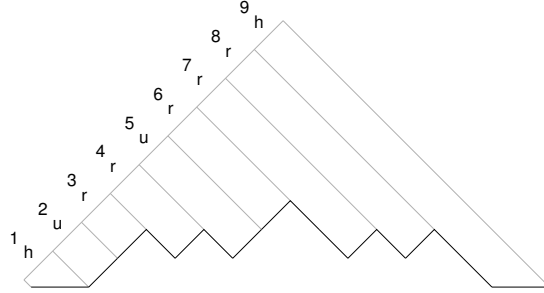
Next we move the r at position 3 down to touch the evolving path, and move the r at position 8 down to meet the path above the d step.



The u at position 4 is moved next but we must insert a d step between positions 6 and 7 to ensure the path is symmetric.



Finally, move the remaining pieces down, inserting **d**'s where appropriate.



Thus we have  $p = \phi^{-1}(\pi) = \text{huududuuddududdh}$ .

**Theorem 2.1.** *The map  $\phi : \text{Sh}_n \rightarrow \mathcal{I}_{n+1}(1234, 1243)$  is a bijection.*

*Proof.* We first show that for any  $p \in \text{Sh}_n$ , the corresponding  $\pi = \phi(p) \in \mathcal{I}_{n+1}(1234, 1243)$ .

Let  $p \in \text{Sh}_n$  and  $p' = p_1 \cdots p_k$  be the corresponding word on the alphabet  $\{u, r, h\}$ . Suppose that  $A_h = \{i_1, \dots, i_\ell\}$ . Then it is clear that  $\pi_{i_j} = i_{\ell+1-j}$  for all  $1 \leq j \leq \ell$ . The same is true for the sets  $A_r$  and  $A_u$  so  $\pi$  is an involution.

From the labelling scheme above, the resulting permutation  $\pi$  has, at most, three levels of right-to-left maxima. It is therefore 1234 avoiding. To show that  $\pi$  is 1243-avoiding, suppose  $\pi$  contains a 1243 pattern  $\pi_i \pi_j \pi_k \pi_\ell$ , where  $\pi_i < \pi_j < \pi_\ell < \pi_k$  and  $i < j < k < \ell$ . The 4 and 3 of the pattern ( $\pi_k$  and  $\pi_\ell$ ) must be in the 1 right-to-left maxima  $A_h$ . Similarly, the 2 (resp. 1) of the pattern must be in the 2 (resp. 3) right-to-left maxima  $A_r$  (resp.  $A_u$ ). Then  $\pi_j \pi_k \pi_\ell$  is a 132-pattern with  $\pi_j \in A_r$  and  $\pi_k, \pi_\ell \in A_h$ . This is not possible for the following reason: Given  $r$  in  $p'$  with label  $r_1$ , let  $h_1, h_2, \dots$  be the sequence of labels of **h**'s to the right of  $r$ . Then  $h_1 > r_1 > h_2, h_3, \dots$ . This statement is easily seen by removing all **u**'s and the suffix **h** from  $p'$  and relabelling. (This relabelling always gives a monotone decreasing sequence.) The fact that the label of the first **h** after  $r$  is greater than the label of the  $r$  is due to the appended **h**.

We now show how to construct the unique path  $p$  corresponding to  $\pi \in \mathcal{I}_{n+1}(1234, 1243)$ . For such a permutation, let  $A_h$ ,  $A_r$  and  $A_u$  be the 1, 2 and 3 right-to-left maxima of  $\pi$ , respectively. Insert **h** at position  $i$  of  $p'$  if  $i \in A_h$  and do likewise for the sets  $A_r$  and  $A_u$ . Remove the suffix **h** from  $p'$  (it is a suffix since  $(n+1)$  is one of the 1 right-to-left maxima). From right to left in  $p'$ , insert a **d** where there is a corresponding **u** and finish by replacing all occurrences of **r** with **ud**. (As was done in the example that preceded the Theorem.) We note

that for each  $p'$  there will be several Schröder paths to which it may correspond, however only one of these is symmetric.  $\square$

From the construction, we also have the following statistics of  $\{1234, 1243\}$ -avoiding involutions:

**Corollary 2.2.** *Let  $p \in \text{Sh}_n$  with  $h$  steps  $h$ ,  $r$  steps  $ud$ , and  $u$  steps  $u$  that are not directly followed by a  $d$  step. Let  $\pi = \phi(p) \in \mathcal{I}_n(1234, 1243)$ .*

- (1) *The number of right-to-left maxima of  $\pi$  is  $h + 1$ .*
- (2) *The number of 2 right-to-left maxima of  $\pi$  is  $r$ .*
- (2) *The number of 3 right-to-left maxima of  $\pi$  is  $u$ .*
- (3) *The number of fixed points of  $\pi$  is  $((1 + h) \bmod 2) + (r \bmod 2) + (u \bmod 2)$ .*

**Open Problem 2.3.** *What statistic on  $\pi = \phi(p)$  corresponds to the height of the path  $p$ ?*

### 3. PROOF OF THEOREM 1.1

To present the proof of Theorem 1.1, we must first consider the enumeration problem for the number  $\mathcal{F}_k$ -avoiding involutions according to length and number of fixed points, where  $\mathcal{F}_k$  is the set of all permutations  $\sigma \in \mathfrak{S}_k$  with  $\sigma_1 = 1$ .

**3.1. Involutions avoiding  $\mathcal{F}_k$ .** In this subsection we present an explicit formula for the number of involutions that avoid all the patterns in  $\mathcal{F}_k$ . To do so we require some new notation. Define  $f_k(n)$  to be the number of involutions  $\pi \in \mathcal{I}_n(\mathcal{F}_k)$ . Given  $t \in [n]$ , we also define

$$f_{k;m}(n; t) = \#\{\pi \in \mathcal{I}_n(\mathcal{F}_k) \mid \pi_1 = t \text{ and } \pi \text{ contains } m \text{ fixed points}\}.$$

Let  $f_k(n; t) = f_k(n, p; t)$  and  $f_k(n) = f_k(n, p)$  be the polynomials  $\sum_{m=0}^n f_{k;m}(n; t)p^m$  and  $\sum_{t=1}^n f_k(n; t)$ , respectively. We denote by  $F_k(x, p)$  the generating function for the sequence  $f_k(n, p)$ , that is  $F_k(x, p) = \sum_{n \geq 0} f_k(n, p)x^n$ .

**Theorem 3.1.** *We have*

$$F_k(x, p) = \sum_{j=0}^{k-2} J_j(p)x^j + \frac{x^{k-1}}{1 - (k-1)x^2}((k-1)J_{k-2}(p)x + J_{k-1}(p)).$$

*Moreover, the number of involutions of length  $k + 2n$  (resp.  $k + 2n - 1$ ) that avoid all the patterns in  $\mathcal{F}_k$  is given by  $(k-1)^{n+1}I_{k-2}$  (resp.  $(k-1)^n I_{k-1}$ ), for all  $n \geq 0$ .*

*Proof.* Let  $\pi \in \mathfrak{S}_n$  be a permutation that avoids all patterns in  $\mathcal{F}_k$ . We have  $\pi_1 \geq n + 2 - k$ . Thus  $\pi \in \mathcal{I}_n(\mathcal{F}_k)$  with  $\pi_1 = t \geq n + 2 - k$  if and only if  $\pi_2 \dots \pi_{t-1} \pi_{t+1} \dots \pi_n$  is an involution on the numbers  $2, \dots, t-1, t+1, \dots, n$  that avoids all the patterns in  $\mathcal{F}_k$ . Hence,  $f_k(n; j) = f_k(n-2)$  for all  $j = n+2-k, n+3-k, \dots, n$ , and  $f_k(n, j) = 0$  for all  $j = 1, 2, \dots, n+1-k$ , where  $n \geq k$ . Thus, for  $n \geq k$ ,

$$f_k(n) = (k-1)f_k(n-2).$$

Using the initial conditions  $f_k(j) = J_j(p)$ ,  $j = 1, 2, \dots, k-1$ , we find that  $f_k(k+2j) = (k-1)^{j+1}J_{k-1}(p)$  and  $f_k(k+2j-1) = (k-1)^j J_{k-2}(p)$  for all  $j \geq 0$ . Rewriting these formulas in terms of generating functions we obtain

$$F_k(x, p) = \sum_{j=0}^{k-2} J_j(p)x^j + \frac{x^{k-1}}{1 - (k-1)x^2}((k-1)J_{k-2}(p)x + J_{k-1}(p)),$$

as claimed.  $\square$

**3.2. Involutions avoiding  $\mathcal{A}_k$ .** In this subsection we prove Theorem 1.1. In order to do this, define  $g_k(n)$  to be the number of involutions  $\pi \in \mathcal{I}_n(\mathcal{A}_k)$  and given  $t_1, t_2, \dots, t_m \in \mathbb{N}$ , we also define

$$g_k(n; t_1, t_2, \dots, t_m) = \#\{\pi_1 \dots \pi_n \in \mathcal{I}_n(\mathcal{A}_k) \mid \pi_1 \dots \pi_m = t_1 \dots t_m\}.$$

**Lemma 3.2.** *Let  $k \geq 3$ . For all  $3 \leq t \leq n+1-k$ ,*

$$g_k(n; t) = (k-2)g_k(n-2; t-1) + \sum_{j=1}^{t-2} g_k(n-2; j),$$

with the initial conditions  $g_k(n; 1) = f_{k-1}(n-1)$ ,  $g_k(n; 2) = f_{k-1}(n-2)$ , and  $g_k(n; t) = g_k(n-2)$  for all  $t = n+2-k, n+3-k, \dots, n$ .

*Proof.* Let  $\pi$  be any involution of length  $n$  that avoids all patterns in  $\mathcal{A}_k$  with  $\pi_1 = t$ . Now let us consider all possible values of  $t$ . If  $t = 1$  then  $\pi \in \mathcal{I}_n(\mathcal{A}_k)$  if and only if  $(\pi_2 - 1)(\pi_3 - 1) \dots (\pi_n - 1) \in \mathcal{I}_{n-1}(\mathcal{F}_{k-1})$ . If  $t = 2$  then  $\pi \in \mathcal{I}_n(\mathcal{A}_k)$  if and only if  $(\pi_3 - 2)(\pi_4 - 2) \dots (\pi_n - 2) \in \mathcal{I}_{n-2}(\mathcal{F}_{k-1})$ . Now assume that  $3 \leq t \leq n+1-k$ , then from the above definitions

$$g_k(n; t) = g_k(n; t, 1) + \dots + g_k(n; t, t-1) + g_k(n; t, t+1) + \dots + g_k(n; t, n).$$

But any involution  $\pi$  satisfying  $\pi_1 < \pi_2 \leq n+2-k$  contains a pattern from the set  $\mathcal{A}_k$  (see the subsequence of the letters  $\pi_1, \pi_2, n+3-k, n+4-k, \dots, n$  in  $\pi$ ). Thus  $g_k(n; t, r) = 0$  for all  $t < r \leq n+2-k$  and so

$$g_k(n; t) = g_k(n; t, 1) + \dots + g_k(n; t, t-1) + g_k(n; t, n+3-k) + \dots + g_k(n; t, n).$$

Also, if  $\pi$  is an involution in  $\mathcal{I}_n$  with  $\pi_1 = t$  and  $\pi_2 = r \geq n+3-k$ , then the entry  $r$  does not appear in any occurrence of  $\tau \in \mathcal{A}_k$  in  $\pi$ . Thus, there exists a bijection between the set of involutions  $\pi \in \mathcal{I}_n(\mathcal{A}_k)$  with  $\pi_1 = t$  and  $\pi_2 = r \geq n+3-k$  and the set of involutions  $\pi' \in \mathcal{I}_{n-2}(\mathcal{A}_k)$  with  $\pi' = t-1$ . Therefore  $g_k(n; t, r) = g_k(n-2; t-1)$  which gives

$$g_k(n; t) = g_k(n; t, 1) + \dots + g_k(n; t, t-1) + (k-2)g_k(n-2; t-1).$$

Also, if  $\pi$  is an involution in  $\mathcal{I}_n$  with  $\pi_1 = t$ ,  $\pi_2 = r < t$  and if  $ta_2 \dots a_k$  is an occurrence of a pattern from the set  $\mathcal{A}_k$  in  $\pi$ , then  $ra_2 \dots a_k$  is an occurrence of a pattern from the set  $\mathcal{A}_k$  in  $\pi$ . Thus, there exists a bijection between the set of involutions  $\pi \in \mathcal{I}_n(\mathcal{A}_k)$  with  $\pi_1 = t$  and  $\pi_2 = r < t$  and the set of involutions  $\pi' \in \mathcal{I}_{n-2}(\mathcal{A}_k)$  with  $\pi'_1 = r-1$ . Therefore  $g_k(n; t, r) = g_k(n-2; r-1)$  which gives

$$g_k(n; t) = (k-2)g_k(n-2; t-1) + \sum_{j=1}^{t-2} g_k(n-2; j),$$

as required. Finally, if  $\pi$  is an involution in  $\mathcal{I}_n$  with  $\pi_1 = t \geq n+2-k$ , then the entry  $t$  does not appear in any occurrence of  $\tau \in \mathcal{A}_k$  in  $\pi$ . Thus, there exists a bijection between the set



of involutions  $\pi \in \mathcal{I}_n(\mathcal{A}_k)$  with  $\pi_1 = t \geq n + 2 - k$  and the set of involutions  $\pi' \in \mathcal{I}_{n-2}(\mathcal{A}_k)$ . Therefore  $g_k(n; t) = g_k(n - 2)$ , as claimed.  $\square$

Let  $G_k(n; v)$  be the polynomial  $\sum_{t=1}^n g_k(n; t)v^{t-1}$ . Rewriting the above lemma in terms of the polynomials  $G_k(n; v)$  we have the following recurrence relation.

**Lemma 3.3.** *Let  $k \geq 3$ . For all  $n \geq k$ ,*

$$\begin{aligned} G_k(n; v) &= f_{k-1}(n-1) + v f_{k-1}(n-2) - v(k-2)f_{k-1}(n-3) + \left(\frac{v^2}{1-v} + (k-2)v\right) G_k(n-2; v) \\ &\quad - \frac{v^n}{1-v} G_k(n-2; 1) + \frac{v^{n-1}}{1-v} \left(k-2 + \frac{v-v^{3-k}}{1-v}\right) G_k(n-4; 1), \end{aligned}$$

where  $G_k(n; v) = I_{n-1} + \frac{v-v^n}{1-v} I_{n-2}$  for all  $n = 0, 1, \dots, k-1$ .

*Proof.* Lemma 3.2 gives

$$\begin{aligned} G_k(n; v) &= f_{k-1}(n-1) + v f_{k-1}(n-2) + \sum_{t=2}^{n-k} v^t \left( (k-2)G_k(n-2; t) + \sum_{j=1}^{t-1} G_k(n-2; j) \right) \\ &\quad + G_k(n-2; 1) \sum_{t=n+1-k}^{n-1} v^t \\ &= f_{k-1}(n-1) + v f_{k-1}(n-2) + \frac{v^2}{1-v} \left( G_k(n-2; v) - G_k(n-4; 1) \sum_{j=n-1-k}^{n-3} v^j \right) \\ &\quad - \frac{v^{n+1-k}}{1-v} (G_k(n-2; 1) - (k-1)G_k(n-4; 1)) + \frac{v^{n+1-k}-v^n}{1-v} G_k(n-2; 1) \\ &\quad + (k-2)v \left( G_k(n-2; v) - f_{k-1}(n-3) - G_k(n-4; 1) \sum_{j=n-k}^{n-3} v^j \right), \end{aligned}$$

which is equivalent to

$$\begin{aligned} G_k(n; v) &= f_{k-1}(n-1) + v f_{k-1}(n-2) - v(k-2)f_{k-1}(n-3) + \left(\frac{v^2}{1-v} + (k-2)v\right) G_k(n-2; v) \\ &\quad - \frac{v^n}{1-v} G_k(n-2; 1) + \frac{v^{n-1}}{1-v} \left(k-2 + \frac{v-v^{3-k}}{1-v}\right) G_k(n-4; 1). \end{aligned}$$

To find the value of  $G_k(n; v)$  for  $n \leq k-1$ , let  $\pi$  be any involution with  $\pi_1 = t$ . If  $t = 1$  then there are  $I_{n-1}$  involutions, whereas if  $t > 1$  there are  $I_{n-2}$  involutions, hence  $G_k(n; v) = v^0 I_{n-1} + \sum_{t=2}^n v^{t-1} I_{n-2} = I_{n-1} + \frac{v-v^n}{1-v} I_{n-2}$ , as required.  $\square$

Lemma 3.3 can be generalised as follows; let  $g_{k;m}(n; t)$  be the number of involutions  $\pi \in \mathcal{I}_n(\mathcal{A}_k)$  such that  $\pi_1 = t$  and  $\pi$  contains exactly  $m$  fixed points. Define  $G_k(n; t; p) = \sum_{m=0}^n g_{k;m}(n; t)p^m$  and  $G_k(n; v; p) = \sum_{t=1}^n G_k(n; t; p)v^{t-1}$ . Using the same arguments as those in the proofs of Lemma 3.2 and Lemma 3.3, while carefully considering the number of fixed points, we have the following result.

**Lemma 3.4.** *Let  $k \geq 3$ . For all  $n \geq k$ ,*

$$\begin{aligned} G_k(n; v, p) &= pf_{k-1}(n-1) + vf_{k-1}(n-2) - pv(k-2)f_{k-1}(n-3) + \left(\frac{v^2}{1-v} + (k-2)v\right) G_k(n-2; v, p) \\ &\quad - \frac{v^n}{1-v} G_k(n-2; 1, p) + \frac{v^{n-1}}{1-v} \left(k-2 + \frac{v-v^{3-k}}{1-v}\right) G_k(n-4; 1, p), \end{aligned}$$

where  $G_k(n; v, p) = pJ_{n-1}(p) + \frac{v-v^n}{1-v} J_{n-2}(p)$  for all  $n = 0, 1, \dots, k-1$ .

Let  $G_k(x, v, p) = \sum_{n \geq 0} G_k(n; v, p)x^n$  be the generating function for the sequence  $G_k(n; v, p)$ . Define  $J_i(v, p)$  to be the polynomial  $\sum d_{tr} v^t p^r$  where  $d_{tr}$  is the number of involutions  $\pi \in \mathcal{I}_i$  such that  $\pi_1 = t+1$  and  $\pi$  contains exactly  $r$  fixed points. Rewriting the recurrence relation in the statement of Lemma 3.4 in terms of generating functions we obtain

$$\begin{aligned} G_k(x, v, p) &= \sum_{j=0}^{k-1} J_j(v, p)x^j + px \left( F_{k-1}(x, p) - \sum_{j=0}^{k-2} J_j(p)x^j \right) + vx^2 \left( F_{k-1}(x, p) - \sum_{j=0}^{k-3} J_j(p)x^j \right) \\ &\quad - (k-2)pvx^3 \left( F_{k-1}(x, p) - \sum_{j=0}^{k-4} J_j(p)x^j \right) - \frac{v^2 x^2}{1-v} \left( G_k(xv, 1, p) - \sum_{j=0}^{k-3} J_j(p)(xv)^j \right) \\ &\quad + vx^2 \left( \frac{v}{1-v} + k-2 \right) \left( G_k(x, v, p) - \sum_{j=0}^{k-3} J_j(v, p)x^j \right) \\ &\quad + \frac{(k-2)v^3 x^4}{1-v} \left( G_k(xv, 1, p) - \sum_{j=0}^{k-5} J_j(p)(xv)^j \right) - \frac{x^4(1-v^{k-2})}{v^{k-6}(1-v)^2} \left( G_k(xv, 1, p) - \sum_{j=0}^{k-5} J_j(p)(xv)^j \right), \end{aligned}$$

which is equivalent to

$$\begin{aligned} \left(1 - \frac{x^2}{1-v} - (k-2)\frac{x^2}{v}\right) G_k(x/v, v, p) &= -\frac{x^2}{1-v} \left(1 - (k-2)\frac{x^2}{v} + \frac{x^2(1-v^{k-2})}{v^{k-2}(1-v)}\right) G_k(x, 1, p) \\ &\quad + \sum_{j=0}^{k-1} J_j(v, p)\frac{x^j}{v^j} + \frac{px}{v} \left( F_{k-1}(x/v, p) - \sum_{j=0}^{k-2} J_j(p)\frac{x^j}{v^j} \right) + \frac{x^2}{v} \left( F_{k-1}(x/v, p) - \sum_{j=0}^{k-3} J_j(p)\frac{x^j}{v^j} \right) \\ &\quad - (k-2)p\frac{x^3}{v^2} \left( F_{k-1}(x/v, p) - \sum_{j=0}^{k-4} J_j(p)\frac{x^j}{v^j} \right) + \frac{x^2}{1-v} \sum_{j=0}^{k-3} J_j(p)x^j \\ &\quad - \frac{x^2}{v} \left( \frac{v}{1-v} + k-2 \right) \sum_{j=0}^{k-3} J_j(v, p)\frac{x^j}{v^j} - \frac{(k-2)x^4}{v(1-v)} \sum_{j=0}^{k-5} J_j(p)x^j + \frac{x^4(1-v^{k-2})}{v^{k-2}(1-v)^2} \sum_{j=0}^{k-5} J_j(p)x^j. \end{aligned}$$

To solve this functional equation, we substitute

$$v := v_0 = \frac{1}{2} \left( 1 + (k-3)x^2 + \sqrt{1 - 2(k-1)x^2 + (k-3)^2 x^4} \right),$$

where  $v_0$  is the root of the coefficient of  $G_k(x/v, v, p)$  above, into the above functional equation, that is,  $1 - \frac{x^2}{1-v_0} - (k-2)\frac{x^2}{v_0} = 0$ . Since  $J_j(v, p) = pJ_{j-1}(p) + \frac{v-v^j}{1-v} J_{j-2}(p)$  for all  $j = 1, 2, \dots, k-1$  and  $J_0(v, p) = 1$ , it is routine to show (via some rather tedious algebraic manipulation) that we obtain Theorem 1.1.

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